Exercise 55

Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

Solution

Start with the definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = \ln x$.

$$(\ln x)' = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$\frac{1}{x} = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

Set $x = 1$.

$$\frac{1}{1} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$1 = \lim_{h \to 0} \frac{\ln(1+h)}{h}$$

Therefore, replacing h with x,

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1.$$